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REDUCIBLE CUBIC FORMS EXPRESSIBLE RATIONALLY AS DETERMINANTS.

By L. E. Dickson.

1. A quadratic form q in three or four variables can be expressed in general in the form $xy - z^2$ or $xy - zw$, each of which is a determinant of order two. Hence if l is any linear form, lq equals a determinant of order three whose elements are linear functions of the variables.

Henceforth, let l and q have rational coefficients. Can we express lq rationally in determinantal form, i.e., as a determinant whose elements are linear functions with rational coefficients? We cannot ordinarily employ the above special method in which the elements of a row are l, 0, 0, since the two linear functions in a row of the minor vanish for rational values, not all zero, of the variables, while q need not vanish for such values. We introduce l as the new variable y. For three variables, yq is always expressible rationally in determinantal form, as shown by taking $w = 0$ in the formula of § 2. For four variables, the question is not so simple, but is answered completely by the following

Theorem. *Let q be a quadratic form in four variables with rational coefficients.* (*i*) *If q vanishes at some rational point having $y = 0$, yq is expressible rationally in determinantal form.* (*ii*) *If $q \neq 0$ for every rational point having $y \neq 0$, then yq is expressible rationally in determinantal form if and only if either yq is equivalent to a ternary form, or the determinant of q is the square of a rational number $\neq 0$ and the determinant of $q(x, 0, z, w)$ is $\neq 0$.* (*iii*) *If both of the preceding hypotheses be denied, so that $q \neq 0$ at every rational point having $y = 0$, and $q = 0$ for some rational point having $y \neq 0$, then yq is not expressible rationally in determinantal form.*

The respective cases are in geometrical language: (*i*) The quadric surface has a rational point in common with the plane. (*ii*) Every rational point of the surface lies in the plane. (*iii*) The surface contains a rational point, but contains no rational point of the plane.

If $q = yL$, yq equals a determinant whose diagonal elements are y, y, L. But if q contains terms free of y, we can apply a linear transformation on x, z, w with rational coefficients which replaces q by a form in which the coefficient of x^2 is $c \neq 0$. We may assume that $c = 1$, since cy may be taken as a new y in yq. After making a suitable addition to x, we obtain yQ, where $Q = x^2 + f$, and f is a quadratic form in y, z, w.

2. First, let Q vanish at a rational point $P = (x', 0, z', w')$ for which $y = 0$. Since z', w' are not both zero, we may take $w' \neq 0$, interchanging z and w if necessary. Taking $zw' - wz'$ as a new variable z, we have $P = (a, 0, 0, 1)$. Let yL denote the sum of the terms of f with the factor y; then

$$Q = x^2 + yL + dz^2 + ezw - a^2w^2, \qquad yQ = \begin{vmatrix} x + aw & y & 0 \\ -L & x - aw & z \\ dz + ew & 0 & y \end{vmatrix}.$$

3. Second, let $Q \neq 0$ for every rational point having $y \neq 0$. Assume that yQ equals a determinant D whose nine elements are linear functions of x, y, z, w with rational coefficients. Since x^2y is the only term involving x in yQ, we may assume that x occurs, with coefficient unity, in the first element of the first row of D and in none of the remaining elements of the first row and first column; also that x occurs, with coefficient unity, in the second element of the second row and not elsewhere in the second row or column; and that the last element of the third row is y. Hence

$$(1) \qquad D = \begin{vmatrix} x + l_1 & l_2 & l_3 \\ l_4 & x - l_1 & l_6 \\ l_7 & l_8 & y \end{vmatrix},$$

where the l's are free of x, while l_3, l_6, l_7, l_8 may be assumed free also of y (in view of the element y), and where the preliminary entry l_5 has been replaced by its value $- l_1$. In fact, the terms linear in x were $xy(l_1 + l_5) - x(l_3l_7 + l_6l_8)$, whence $l_1 + l_5 \equiv 0$ and

$$(2) \qquad l_3l_7 + l_6l_8 \equiv 0.$$

Since $D = 0$ when $x = - l_1$, $l_2 = l_3 = 0$ (which are satisfied by an infinitude of rational values of x, y, z, w), these linear relations must imply $y = 0$, in view of our hypothesis that $yQ \neq 0$ if $y \neq 0$. Hence y equals a linear homogeneous function of l_2 and l_3. But l_3 is free of y. Hence

$$(3) \qquad l_2 \equiv \rho y + \sigma l_3, \qquad \rho \neq 0.$$

Using similarly the elements of the second row, first and second columns, we see that

$$(4) \qquad l_4 \equiv ry + tl_6, \qquad r \neq 0,$$

that l_4 is a linear function of y and l_7, and that l_2 is a linear function of y and l_8. Thus, if $\sigma l_3 \not\equiv 0$,

$$(5) \qquad l_8 = vl_3, \qquad l_7 = - vl_6, \qquad v \neq 0,$$

by (2); the same follow also if $tl_6 \not\equiv 0$. By (5),

$$D = y(x^2 - l_1^2 - l_2l_4) - v\lambda, \qquad \lambda \equiv 2l_1l_3l_6 + l_2l_6^2 - l_3^2l_4.$$

By (3) and (4),

$$\lambda = l_3 l_6(2l_1 + \sigma l_6 - t l_3) + \rho y l_6^2 - r y l_3^2.$$

Since D shall have the factor y, we conclude that

$$l_1 \equiv \tfrac{1}{2} t l_3 - \tfrac{1}{2} \sigma l_6 \tag{6}$$

if $l_3 l_6 \not\equiv 0$ and if l_1 (like l_3 and l_6) is free of y. This is accomplished as follows. Add the products of the elements of the first row of D by k to the elements of the second row, and then subtract the products of the elements of the second column by k from the elements of the first column. We obtain a determinant of the same form as (1) with $l_1 - k l_2$ in place of l_1. By choice of k, we may assume that l_1 lacks y. We now have

$$Q = D/y = x^2 - l_1^2 - l_2 l_4 - v \rho l_6^2 + v r l_3^2. \tag{7}$$

Inserting the values (3), (4), (6) of l_2, l_4, l_1, we obtain a quadratic form in x, y, l_3, l_6, whose determinant equals $(r\rho m)^2$, where

$$m = v - \frac{t^2}{4r} + \frac{\sigma^2}{4\rho},$$

while the determinant of the part in l_3, l_6 only is $-v r \rho m$. Or we may avoid this computation by completing the square of the terms in y and finding that the terms in $l_3 l_6$ cancel:

$$Q = x^2 - r\rho Y^2 + r m l_3^2 - \rho m l_6^2, \qquad Y = y + \frac{\sigma}{2\rho} l_3 + \frac{t}{2r} l_6.$$

The determinant of Q is seen by inspection to be $(r\rho m)^2$. If $m = 0$, then $Q = 0$ when $x = Y = 0$, which imply $y = 0$ only when $Y \equiv y$, and then yQ is a binary form.

It remains to consider the special cases excluded above. If $l_3 \equiv l_6 \equiv 0$,

$$Q = D/y = x^2 - l_1^2 - r\rho y^2 \tag{8}$$

vanishes when $x - l_1 = \alpha y$, $x + l_1 = \beta y$, $\alpha\beta = r\rho$, $\alpha \neq \beta$, which imply $y = 0$ only when $l_1 \equiv 0$, and then yQ is a binary form.

If $l_3 \equiv 0$, $l_6 \not\equiv 0$, then $l_8 \equiv 0$ by (2). Using (3) and (4), we get

$$Q = D/y = x^2 - l_1^2 - \rho y(r y + t l_6) + \rho l_6 l_7.$$

Now $Q = 0$ when $x = l_1$, $y = l_6$, $ry + t l_6 = l_7$, which imply $y = 0$ only when l_7 is proportional to l_6. Hence let $l_7 \equiv k l_6$. Then $Q = 0$ when $l_6 = y$, $x - l_1 = \alpha y$, $x + l_1 = \beta y$, $\alpha\beta = \rho(r + t - k)$, $\alpha \neq \beta$, which imply $y = 0$ only when $l_1 \equiv 0$, and then yQ is a ternary form in x, y, l_6.

If $l_3 \not\equiv 0$, $l_6 \equiv 0$, we interchange the first two rows and first two columns of (1) and are led to the preceding case.

Finally, let $l_3 l_6 \not\equiv 0$. By the remarks accompanying (5), it remains to consider only the case in which $\sigma = t = 0$, whence $l_2 = \rho y$, $l_4 = ry$. By (2), l_6 or l_8 is divisible by l_3. In the second alternative, we have (5), since, if $l_8 \equiv 0$, then $l_7 \equiv 0$ and D is of the form (8). Hence $l_6 \equiv s l_3$, $s \neq 0$, so that $l_7 \equiv -s l_8$ by (2). The only term of D lacking y is $-2 s l_1 l_3 l_8$, whence $l_1 l_8 \equiv 0$. But $l_8 \equiv 0$ was seen to lead to (8). Hence $l_1 \equiv 0$ and

$$D/y = x^2 - r\rho y^2 + (r - s^2\rho) l_3 l_8$$

is zero when $x = r$, $y = s$, $l_3 l_8 = -r$. But l_3 and l_8 can be given any desired values by choice of z and w unless they are proportional, which is the case (5) already treated.

Conversely, let the determinant of $x^2 + f$ be a rational square $\neq 0$ and the determinant of $f(0, z, w)$ be $\neq 0$. By a linear transformation altering neither x nor y and having rational coefficients we can evidently delete the terms in zw, yz, yw. We obtain a form of the following type, whose simplest representation as a determinant is obtained by taking $\sigma = t = 0$ in (1)–(7):

$$y(x^2 - r\rho y^2 + rv l_3^2 - \rho v l_6^2) \equiv \begin{vmatrix} x & \rho y & l_3 \\ ry & x & l_6 \\ -v l_6 & v l_3 & y \end{vmatrix}.$$

4. If Q falls under neither § 2 nor § 3, then $Q \neq 0$ for every rational point having $y = 0$, and $Q = 0$ for some rational point P with $y \neq 0$, say $P = (x', 1, z', w')$. We shall prove* that yQ is not equal to a determinant (1) with rational elements. Taking $z - z'y$ and $w - w'y$ as new variables z, w, we may write $P = (a, 1, 0, 0)$. In view of (2), the expansion of (1) gives

$$(9) \qquad D = y(x^2 - l_1^2 - l_2 l_4) + l_3 l_4 l_8 + l_2 l_6 l_7 + 2 l_1 l_3 l_7.$$

First, let one of l_2, l_4 contain y. Interchanging rows and columns if necessary, we may assume that

$$l_2 = \rho y + L_2, \qquad l_4 = ry + L_4, \qquad \rho \neq 0,$$

where L_2, L_4 are functions of z, w. By the argument just above (7) in § 3, we may assume that l_1 lacks y. Since (9) shall equal yQ,

$$Q = x^2 - l_1^2 - l_2 l_4 + r l_3 l_8 + \rho l_6 l_7, \qquad R \equiv l_3 l_8 L_4 + l_6 l_7 L_2 + 2 l_1 l_3 l_7 \equiv 0.$$

If $l_3 \equiv l_6 \equiv 0$, $Q = 0$ when $x = l_1$, $y = 0$, $L_2 = 0$, contrary to hypothesis. If $l_3 \equiv 0$, $l_6 \not\equiv 0$, then $l_8 \equiv 0$ by (2), $l_7 L_2 \equiv 0$ by $R \equiv 0$, and $Q = 0$ when $x = l_1$, $y = 0$, $L_2 L_4 = \rho l_6 l_7$, which implies a linear relation between z and w. Hence $l_3 \not\equiv 0$, and, similarly, $l_6 \not\equiv 0$, $l_7 \not\equiv 0$, $l_8 \not\equiv 0$.

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* The hypotheses are satisfied if $Q = x^2 - y^2 + 2z^2 + 3w^2$, $P = (1, 1, 0, 0)$.

By (2), l_6 or l_8 is divisible by l_3. In the first case,

$$(10) \qquad l_6 \equiv \alpha l_3, \qquad l_7 \equiv -\alpha l_8, \qquad \alpha \neq 0,$$

whence $2\alpha l_1 \equiv L_4 - \alpha^2 L_2$ by $R \equiv 0$. Then $Q = 0$ for

$$y = 0, \qquad x = \frac{L_4 + \alpha^2 L_2}{2\alpha}, \qquad l_3 = 0,$$

contrary to hypothesis. Hence l_6 and l_3 are not proportional and

$$(11) \qquad l_8 \equiv \beta l_3, \qquad l_7 \equiv -\beta l_6, \qquad \beta \neq 0,$$

whence $l_3^2 L_4 - l_6^2 L_2 - 2l_1 l_3 l_6 \equiv 0$ by $R \equiv 0$. Thus $L_4 \equiv k l_6$ and $k l_3^2 - l_6 L_2 - 2 l_1 l_3 \equiv 0$, whence $L_2 \equiv p l_3$, $2l_1 \equiv k l_3 - p l_6$. Thus

$$Q = x^2 - \tfrac{1}{4}(k l_3 - p l_6)^2 - (\rho y + p l_3)(r y + k l_6) + r\beta l_3^2 - \rho\beta l_6^2.$$

Since Q is zero at P, $\rho r = a^2$, whence $r = \rho t^2$, $t = a/\rho$. Thus

$$Q_{y=0} = x^2 - M^2 + \beta\rho(t^2 l_3^2 - l_6^2), \qquad M \equiv \tfrac{1}{2}(k l_3 + p l_6),$$

so that $Q = 0$ for $y = 0$, $x = M$, $l_6 = t l_3$, contrary to hypothesis.

Second, let l_2 and l_4 both lack y. Let $l_1 = cy + L_1$. Then, by (9),

$$Q = x^2 - l_1^2 - l_2 l_4 + 2c l_3 l_7, \qquad S \equiv l_3 l_4 l_8 + l_2 l_6 l_7 + 2L_1 l_3 l_7 \equiv 0.$$

If $l_3 l_7 \equiv 0$, $Q = 0$ when $x = L_1$, $y = 0$, $l_2 = 0$, contrary to hypothesis. Thus $l_3 l_7 \not\equiv 0$ and, similarly, $l_2 l_4 \not\equiv 0$. Then $l_6 l_8 \not\equiv 0$ by (2). We have (10) or (11). By (10), $S \equiv 0$ implies $2\alpha L_1 \equiv l_4 - \alpha^2 l_2$. Then $Q = 0$ when

$$y = 0, \qquad l_3 = 0, \qquad x = (l_4 + \alpha^2 l_2)/(2\alpha).$$

By (11), $S \equiv \beta(l_3^2 l_4 - l_2 l_6^2 - 2L_1 l_3 l_6) \equiv 0$. Hence $l_2 \equiv d l_3$ and then $l_4 \equiv e l_6$, $2L_1 \equiv e l_3 - d l_6$. Thus $Q = 0$ when $y = l_3 = 0$, $x = \tfrac{1}{2} d l_6$, contrary to hypothesis.